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Effect of streamwise vortices on Tollmien–Schlichting waves

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The method of multiple scales is used to determine a first-order uniform expansion for the effect of counter-rotating steady streamwise vortices in growing boundary layers on oblique Tollmien–Schlichting waves. The results show that such vortices have a strong tendency to amplify oblique Tollmien–Schlichting waves having a spanwise wavelength that is twice the wavelength of the vortices. An analytical expression is derived for the growth rates of these waves. These exponential growth rates increase linearly with increasing amplitudes of the vortices. Numerical results are presented. They suggest that this mechanism may dominate the instability.

1. Introduction

We consider the effect of counter-rotating steady streamwise vortices on the instability of growing boundary layers. We describe a parametric instability mechanism by which such vortices amplify selected oblique Tollmien–Schlichting waves. To first order, the selected waves have a spanwise wavelength that is twice that of the vortices.

Weak and moderately strong steady streamwise vortices arise abundantly in boundary layers from many causes. In a series of wind-tunnel tests over flat plates with zero-pressure gradient, Klebanoff & Tidstrom (1959) observed steady quasi-periodic variations in the spanwise direction (streamwise vortices) evidently evoked by free-stream conditions. Similar vortices were observed in a National Physical Laboratory tunnel specifically designed for the study of two-dimensional boundary layers. Bradshaw (1965) found that these variations may appear downstream of slightly non-uniform settling-chamber damping screens, depending on their solidity. Using the method of matched asymptotic expansions, Crow (1966) inferred the effect of a small, periodic incident transverse flow on the mean boundary layer over a flat plate.

Görtler (1941) found that a boundary layer over a concave surface is strongly unstable. The instability is manifested by the presence of counter-rotating vortices (called Görtler vortices) having their axis in the streamwise direction. Using the china-clay technique, Gregory & Walker (1956) were the first to observe traces of Görtler vortices. Then, Aihara (1962) and Tani & Sakagami (1962) used coloured liquids and smoke threads to visualize these vortices. Subsequently, Wortmann (1964) used the tellurium method to visualize these vortices in a water tunnel. Then, Bippes (1978) and Bippes & Görtler (1972) conducted experiments on walls with the radii of curvature 0·5 and 1 m so that the generated Görtler vortices were fairly strong. They made the flow visible by using the hydrogen-bubble technique and photographed it with a photogrammatic stereocamera. The photographs were analysed photogrammetrically and fairly accurate quantitative information of the flow field was obtained. Using

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hot-wire measurements, Aihara (1962) and Tani (1961, 1962) found three-dimensional counter-rotating vortices with spanwise vorticity in a boundary layer over a concave wall. Unlike the case of pre-existing streamwise vortices, Görtler vortices generated by a concave surface are amplified with streamwise distance. Their amplification is exponential (Smith 1955) when they are weak and it appears to be linear when they are strong (Bippes 1978).

These vortices by themselves may not lead to the transition of laminar flows to turbulent flows. The influence of steady streamwise vortices on two-dimensional Tollmien–Schlichting waves was studied experimentally by Tani (1961), Aihara (1962, 1976), Tani & Sakagami (1962), Tani & Aihara (1969), Bippes (1978), and Wortmann (1969). Aihara (1962) and Tani & Aihara (1969) concluded that the Görtler vortices indirectly affect the transition by inducing a spanwise variation in the boundary-layer thickness, at least when the radii of curvature are not extremely small. Tani (1961) found the spatial amplification of the Görtler vortices to be small, even at downstream locations close to the transition point. However, these vortices deform the mean flow field and induce a spanwise variation in the boundary-layer thickness, resulting in the development of velocity profiles having varying stability characteristics along the span. The modification of the mean flow modifies the amplification of the unsteady waves. Wortmann (1969) observed a secondary steady instability following the appearance of the Görtler vortices before three-dimensional unstable waves that lead to transition set in. Bippes (1978) and Aihara (1976) observed meandering or pulsating vortices before turbulence sets in. Thus, available experimental evidence indicates that the Görtler vortices do not lead directly to turbulence without a coupling with unsteady two- or three-dimensional unstable disturbances.

The above shows that there are many theoretical and experimental studies relating to the generation of streamwise vortices and a number of experimental studies relating to their effect on transition, but to the author's knowledge, no theory yet exists on how these vortices affect the development of Tollmien–Schlichting waves. The purpose of the present paper is to present a parametric instability mechanism by which the streamwise vortices (for definiteness, Görtler vortices over curved surfaces) increase the growth of selected oblique Tollmien–Schlichting waves in growing boundary layers. To first order, the selected waves have a spanwise wavelength that is twice that of the vortices and their growth rates may increase by a factor of four or five.

2. Problem formulation

We consider the stability of a basic flow that consists of the superposition of a steady two-dimensional boundary layer and a flow corresponding to growing steady quasi-periodic counter-rotating streamwise vortices. For definiteness, we assume that the vortices are Görtler vortices resulting from the instability of a boundary layer over a two-dimensional concave surface.

We employ an orthogonal curvilinear body-oriented co-ordinate system x, y, z such that x measures distances along the curved wall, y measures distances normal to the surface, and z is a rectilinear co-ordinate normal to x and y. We introduce dimensionless quantities using the freestream velocity U_∞ and the displacement thickness δ_r at x_r so that the Reynolds number is given by $R = U_\infty \delta_r/\nu$, where ν is the kinematic viscosity which is assumed to be constant.

The boundary layer is assumed to be slightly non-parallel so that the flow field is a slowly varying function of the streamwise position x. To express this slow variation, we introduce the scale $x_1 = \epsilon x$, where $\epsilon = R^{-1}$ is a small dimensionless parameter that characterizes the nonparallelism of the boundary layer; $\epsilon = 0$ for truly parallel flows. Using x_1, we express the boundary-layer pressure and streamwise and normal velocity components as $P_0(x_1)$, $U_0(x_1, y)$, and $\epsilon V_0(x_1, y)$, respectively.

Görtler (1941) was the first to show that the above boundary-layer flow over a concave surface is strongly unstable. The instability takes the form of counter-rotating streamwise vortices called Görtler vortices. Floryan & Saric (1979) and Ragab & Nayfeh (1980) gave a comprehensive review of the different analyses of these vortices. Following Floryan & Saric and Ragab & Nayfeh and using a modified notation, we write the disturbance representing the Görtler vortices for the case of a quasi-parallel flow as

$$u = \tilde{U}_1(x_1, y)\cos 2\beta z \exp\left[\int \sigma\, dx_1\right], \tag{1}$$

$$v = R^{-1}\tilde{V}_1(x_1, y)\cos 2\beta z \exp\left[\int \sigma\, dx_1\right], \tag{2}$$

$$w = R^{-1}\tilde{W}_1(x_1, y)\sin 2\beta z \exp\left[\int \sigma\, dx_1\right], \tag{3}$$

$$p = R^{-2}\tilde{P}_1(x_1, y)\cos 2\beta z \exp\left[\int \sigma\, dx_1\right]. \tag{4}$$

For the case of zero pressure gradient,

$$D^2\tilde{U}_1 - V_0 D\tilde{U}_1 + (H_0 - 4\beta^2 - \sigma U_0)\tilde{U}_1 - E_0\tilde{V}_1 = 0, \tag{5}$$

$$D^2\tilde{V}_1 - V_0 D\tilde{V}_1 + (-H_0 - 4\beta^2 - \sigma U_0)\tilde{V}_1 - (2U_0 G_N^2 + G_0)\tilde{U}_1 - D\tilde{P}_1 = 0, \tag{6}$$

$$D^2\tilde{W}_1 - V_0 D\tilde{W}_1 + (-4\beta^2 - \sigma U_0)\tilde{W}_1 + 2\beta\tilde{P}_1 = 0, \tag{7}$$

$$D\tilde{V}_1 + \sigma\tilde{U}_1 + 2\beta\tilde{W}_1 = 0, \tag{8}$$

$$\tilde{U}_1 = \tilde{V}_1 = \tilde{W}_1 = 0 \quad \text{at} \quad y = 0, \tag{9}$$

$$\tilde{U}_1, \tilde{V}_1, \tilde{W}_1 \to 0 \quad \text{as} \quad y \to \infty, \tag{10}$$

where $D = d/dy$; G_N is the Görtler number defined by

$$G_N^2 = R^2 K \tag{11}$$

(where K surface curvature) and

$$E_0 = \frac{1}{\sqrt{2}}\frac{\partial U_0}{\partial y}, \quad H_0 = \frac{1}{\sqrt{2}}\frac{\partial V_0}{\partial y}, \quad G_0 = -\tfrac{1}{2}V_0 - \frac{1}{\sqrt{2}}yH_0. \tag{12}$$

For the case of flows with pressure gradients, the body-oriented co-ordinate system is not suitable for studying the Görtler instability because V_0 grows linearly with y as $y \to \infty$. Ragab & Nayfeh (1980) showed that V_0 is bounded as $y \to \infty$ in a co-ordinate system based on the potential lines and streamlines of the inviscid flow. Moreover, they showed that the pressure gradient affects the Görtler instability through the modification of the stability through equations (5)–(8) as well as the mean flow.

It follows from (1)–(4) that V_1, W_1, and P_1 are much smaller than U_1 and hence they can be neglected for the case of Görtler vortices. In order that the analysis be applicable to the case of general streamwise vortices, we include them in the analysis. For the purpose of the present study, we express (1)–(4) as

$$u = U_1(x_1, y)\cos 2\beta z, \quad v = V_1(x_1, y)\cos 2\beta z, \tag{13}$$

$$w = W_1(x_1, y)\sin 2\beta z, \quad p = P_1(x_1, y)\cos 2\beta z, \tag{14}$$

We normalize $U_1(x_1, y)$ in (13) so that its maximum value is unity. As mentioned earlier, we consider the stability of a basic flow that consists of the superposition of the boundary-layer flow described by the subscript 0 and a flow corresponding to Görtler vortices. If the amplitude of the vortices (i.e., ratio of maximum streamwise velocity component to free-stream velocity) is ϵ_v, the basic flow to be studied is given by

$$U = U_0(x_1, y) + \epsilon_v U_1(x_1, y)\cos 2\beta z + \dots, \tag{15}$$

$$V = \epsilon V_0(x_1, y) + \epsilon_v V_1(x_1, y)\cos 2\beta z + \dots, \tag{16}$$

$$W = \epsilon_v W_1(x_1, y)\sin 2\beta z + \dots, \tag{17}$$

$$P = P_0(x_1) + \epsilon_v P_1(x_1, y)\cos 2\beta z + \dots. \tag{18}$$

We study the stability of this basic flow to oblique Tollmien–Schlichting waves. To this end, we superpose the small unsteady perturbation quantities $\epsilon_T u(x,y,z,t)$, $\epsilon_T v(x,y,z,t)$, $\epsilon_T w(x,y,z,t)$ and $\epsilon_T p(x,y,z,t)$ on those given in (15)–(18) so that the total flow quantities become $U+\epsilon_T u$, $V+\epsilon_T v$, $W+\epsilon_T w$, and $P+\epsilon_T p$. Here, ϵ_T is a small dimensionless quantity that is the order of the amplitude of the Tollmien–Schlichting waves. In this paper, ϵ_T is assumed to be much smaller than ϵ_v and ϵ so that terms the order of ϵ_T^2 can be neglected compared with $\epsilon_T \epsilon_v$ and $\epsilon_T \epsilon$. Substituting these total flow quantities into the dimensionless Navier–Stokes equations, subtracting the basic-flow quantities, and keeping linear terms in ϵ_T, we obtain

$$\frac{u}{x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{19}$$

$$\frac{\partial u}{\partial t} + U\frac{\partial u}{\partial x} + u\frac{\partial U}{\partial x} + V\frac{\partial u}{\partial y} + v\frac{\partial U}{\partial y} + W\frac{\partial u}{\partial z} + w\frac{\partial U}{\partial z} = -\frac{\partial p}{\partial x} + \frac{1}{R}\nabla^2 u, \tag{20}$$

$$\frac{\partial v}{\partial t} + U\frac{\partial v}{\partial x} + u\frac{\partial V}{\partial x} + V\frac{\partial v}{\partial y} + v\frac{\partial V}{\partial y} + W\frac{\partial v}{\partial z} + w\frac{\partial V}{\partial z} = -\frac{\partial p}{\partial y} + \frac{1}{R}\nabla^2 v, \tag{21}$$

$$\frac{\partial w}{\partial t} + U\frac{\partial w}{\partial x} + u\frac{\partial W}{\partial x} + V\frac{\partial w}{\partial y} + v\frac{\partial W}{\partial y} + W\frac{\partial w}{\partial z} + w\frac{\partial W}{\partial z} = -\frac{\partial p}{\partial z} + \frac{1}{R}\nabla^2 w, \tag{22}$$

where t is made dimensionless by using δ_r/U_∞.

Substituting (15)–(18) into (19)–(22), we obtain

$$\frac{\partial u}{\partial t} + U_0\frac{\partial u}{\partial x} + v\frac{\partial U_0}{\partial y} + \frac{\partial p}{\partial x} - \frac{1}{R}\nabla^2 u = -\epsilon_v\left\{\left[U_1\frac{\partial u}{\partial x} + V_1\frac{\partial u}{\partial y} + \frac{\partial U_1}{\partial y}v\right]\cos 2\beta z\right.$$
$$\left. + \left[W_1\frac{\partial u}{\partial z} - 2\beta U_1 w\right]\sin 2\beta z\right\} - \epsilon\left[\frac{\partial U_0}{\partial x_1}u + V_0\frac{\partial u}{\partial y}\right] + O(\epsilon\epsilon_v), \tag{23}$$

$$\frac{\partial v}{\partial t}+U_0\frac{\partial v}{\partial x}+\frac{\partial p}{\partial y}-\frac{1}{R}\nabla^2 v = -\epsilon_v\left\{\left[U_1\frac{\partial v}{\partial x}+V_1\frac{\partial v}{\partial y}+\frac{\partial V_1}{\partial y}v\right]\cos 2\beta z\right.$$
$$\left.+\left[W_1\frac{\partial v}{\partial z}-2\beta V_1 w\right]\sin 2\beta z\right\}-\epsilon\left[V_0\frac{\partial v}{\partial y}+\frac{\partial V_0}{\partial y}v\right]+O(\epsilon^2,\epsilon\epsilon_v), \quad (24)$$

$$\frac{\partial w}{\partial t}+U_0\frac{\partial w}{\partial x}+\frac{\partial p}{\partial z}-\frac{1}{R}\nabla^2 w = -\epsilon_v\left\{\left[U_1\frac{\partial w}{\partial x}+V_1\frac{\partial w}{\partial y}+2\beta W_1 w\right]\cos 2\beta z\right.$$
$$\left.+\left[\frac{\partial W_1}{\partial y}v+W_1\frac{\partial w}{\partial z}\right]\sin 2\beta z\right\}-\epsilon V_0\frac{\partial w}{\partial y}+O(\epsilon\epsilon_v). \quad (25)$$

Equations (19) and (23)–(25) need to be supplemented by initial and boundary conditions. The initial conditions are specified later, whereas the boundary conditions for an impermeable flat surface are

$$u = v = w = 0 \quad \text{at} \quad y = 0, \quad (26)$$

$$u, v, w \to 0 \quad \text{as} \quad y \to \infty. \quad (27)$$

3. Solution

We use the method of multiple scales (e.g., Nayfeh 1973) to determine a first-order uniform expansion for (19) and (23)–(27). To accomplish this, we let $\epsilon_v = O(\epsilon)$ and write $\epsilon_v = \chi\epsilon$, where $\chi = O(1)$. If $\epsilon \ll \epsilon_v$, the effect of the growth of the boundary layer is small compared with the effect of the vortices. If $\epsilon_v \ll \epsilon$, the effect of the vortices is small compared with that due to the growth of the boundary layer, and the solution accounts for the non-parallel effects only. Thus, the above ordering yields an expansion that accounts for the effects of the streamwise vortices and the growth of the boundary layer, and it includes the cases $\epsilon \ll \epsilon_v$ and $\epsilon_v \ll \epsilon$ as special cases.

We seek a uniform expansion for (19) and (23)–(27) in the form

$$u = \sum_{n=0}^{1} \epsilon^n u_n(x_0, x_1, y, z_0, z_1, t_0, t_1) + O(\epsilon^2), \quad (28)$$

$$v = \sum_{n=0}^{1} \epsilon^n v_n(x_0, x_1, y, z_0, z_1, t_0, t_1) + O(\epsilon^2), \quad (29)$$

$$w = \sum_{n=0}^{1} \epsilon^n w_n(x_0, x_1, y, z_0, z_1, t_0, t_1) + O(\epsilon^2), \quad (30)$$

$$p = \sum_{n=0}^{1} \epsilon^n p_n(x_0, x_1, y, z_0, z_1, t_0, t_1) + O(\epsilon^2), \quad (31)$$

where

$$x_n = \epsilon^n x, \quad z_n = \epsilon^n z, \quad t_n = \epsilon^n t. \quad (32)$$

Substituting (28)–(32) into (19) and (23)–(27) and equating coefficients of like powers of ϵ, we obtain the following:

Order ϵ^0:

$$\mathscr{L}_1(u_0, v_0, w_0) = \frac{\partial u_0}{\partial x_0}+\frac{\partial v_0}{\partial y}+\frac{\partial w_0}{\partial z_0} = 0, \quad (33)$$

$$\mathscr{L}_2(u_0, v_0, p_0) = \frac{\partial u_0}{\partial t_0}+U_0\frac{\partial u_0}{\partial x_0}+v_0\frac{\partial U_0}{\partial y}+\frac{\partial p_0}{\partial x_0}-\frac{1}{R}\nabla_0^2 u_0 = 0, \quad (34)$$

$$\mathscr{L}_3(v_0, p_0) = \frac{\partial v_0}{\partial t_0} + U_0 \frac{\partial v_0}{\partial x_0} + \frac{\partial p_0}{\partial y} - \frac{1}{R} \nabla_0^2 v_0 = 0, \tag{35}$$

$$\mathscr{L}_4(w_0, p_0) = \frac{\partial w_0}{\partial t_0} + U_0 \frac{\partial w_0}{\partial x_0} + \frac{\partial p_0}{\partial z_0} - \frac{1}{R} \nabla_0^2 w_0 = 0, \tag{36}$$

$$u_0 = v_0 = w_0 = 0 \quad \text{at} \quad y = 0, \tag{37}$$

$$u_0, v_0, w_0 \to 0 \quad \text{as} \quad y \to \infty. \tag{38}$$

Order ϵ:

$$\mathscr{L}_1(u_1, v_1, w_1) = -\frac{\partial u_0}{\partial x_1} - \frac{\partial w_0}{\partial z_1}, \tag{39}$$

$$\begin{aligned}\mathscr{L}_2(u_1, v_1, p_1) = {} & -\frac{\partial u_0}{\partial t_1} - U_0 \frac{\partial u_0}{\partial x_1} - \frac{\partial p_0}{\partial x_1} + \frac{2}{R}\left[\frac{\partial^2 u_0}{\partial x_0 \partial x_1} + \frac{\partial^2 u_0}{\partial z_0 \partial z_1}\right] \\ & - \chi\left\{\left[U_1 \frac{\partial u_0}{\partial x_0} + V_1 \frac{\partial u_0}{\partial y} + \frac{\partial U_1}{\partial y} v_0\right] \cos 2\beta z_0 + \left[W_1 \frac{\partial u_0}{\partial z_0} - 2\beta U_1 w_0\right] \sin 2\beta z_0\right\} \\ & - \frac{\partial U_0}{\partial x_1} u_0 - V_0 \frac{\partial u_0}{\partial y},\end{aligned} \tag{40}$$

$$\begin{aligned}\mathscr{L}_3(v_1, p_1) = {} & -\frac{\partial v_0}{\partial t_1} - U_0 \frac{\partial v_0}{\partial x_1} + \frac{2}{R}\left[\frac{\partial^2 v_0}{\partial x_0 \partial x_1} + \frac{\partial^2 v_0}{\partial z_0 \partial z_1}\right] - \chi\left\{\left[U_1 \frac{\partial v_0}{\partial x_0} + V_1 \frac{\partial v_0}{\partial y} + \frac{\partial V_1}{\partial y} v_0\right] \cos 2\beta z_0\right. \\ & \left. + \left[W_1 \frac{\partial v_0}{\partial z_0} - 2\beta V_1 w_0\right] \sin 2\beta z_0\right\} - V_0 \frac{\partial v_0}{\partial y} - \frac{\partial V_0}{\partial y} v_0,\end{aligned} \tag{41}$$

$$\begin{aligned}\mathscr{L}_4(w_1, p_1) = {} & -\frac{\partial w_0}{\partial t_1} - U_0 \frac{\partial w_0}{\partial x_1} - \frac{\partial p_0}{\partial z_1} + \frac{2}{R}\left[\frac{\partial^2 w_0}{\partial x_0 \partial x_1} + \frac{\partial^2 w_0}{\partial z_0 \partial z_1}\right] \\ & - \chi\left\{\left[U_1 \frac{\partial w_0}{\partial x_0} + V_1 \frac{\partial w_0}{\partial y} + 2\beta W_1 w_0\right] \cos 2\beta z_0 + \left[\frac{\partial W_1}{\partial y} v_0 + W_1 \frac{\partial w_0}{\partial z_0}\right] \sin 2\beta z_0\right\} - V_0 \frac{\partial w_0}{\partial y},\end{aligned} \tag{42}$$

$$u_1 = v_1 = w_1 = 0 \quad \text{at} \quad y = 0, \tag{43}$$

$$u_1, v_1, w_1 \to 0 \quad \text{as} \quad y \to \infty. \tag{44}$$

In the above

$$\nabla_0^2 = \frac{\partial^2}{\partial x_0^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z_0^2}.$$

The initial conditions are taken such that the solution of the zeroth-order problem, (33)–(38), consists of two wave packets centred around the frequency ω, the streamwise wavenumber α, and the spanwise wavenumbers β_1 and $-\beta_1$; that is,

$$u_0 = A_1(x_1, z_1, t_1)\, \zeta_{11}(y, x_1) \exp(i\theta_1) + A_2(x_1, z_1, t_1)\, \zeta_{21}(y, x_1) \exp(i\theta_2), \tag{45}$$

$$v_0 = A_1 \zeta_{12}(y, x_1) \exp(i\theta_1) + A_2 \zeta_{22}(y, x_1) \exp(i\theta_2), \tag{46}$$

$$w_0 = A_1 \zeta_{13}(y, x_1) \exp(i\theta_1) + A_2 \zeta_{23}(y, x_1) \exp(i\theta_2), \tag{47}$$

$$p_0 = A_1 \zeta_{14}(y, x_1) \exp(i\theta_1) + A_2 \zeta_{24}(y, x_1) \exp(i\theta_2), \tag{48}$$

where

$$\theta_n = \int \alpha(x_1)\, dx - \beta_n z_0 - \omega t_0, \quad \beta_2 = -\beta_1, \tag{49}$$

and the functions A_1 and A_2 are undetermined at this level of approximation; they are determined by imposing the solvability conditions at the next level of approximation. Substituting (45)–(49) into (33)–(38) yields the following eigenvalue problems:

$$i\alpha\zeta_{n_1}+D\zeta_{n_2}-i\beta_n\zeta_{n_3}=0, \tag{50}$$

$$i(U_0\alpha-\omega)\zeta_{n_1}+\zeta_{n_2}DU_0+i\alpha\zeta_{n_4}-\frac{1}{R}(D^2-\alpha^2-\beta_1^2)\zeta_{n_1}=0, \tag{51}$$

$$i(U_0\alpha-\omega)\zeta_{n_2}+D\zeta_{n_4}-\frac{1}{R}(D^2-\alpha^2-\beta_1^2)\zeta_{n_2}=0, \tag{52}$$

$$i(U_0\alpha-\omega)\zeta_{n_3}-i\beta_n\zeta_{n_4}-\frac{1}{R}(D^2-\alpha^2-\beta_1^2)\zeta_{n_3}=0, \tag{53}$$

$$\zeta_{n_1}=\zeta_{n_2}=\zeta_{n_3}=0 \quad \text{at} \quad y=0, \tag{54}$$

$$\zeta_{n_1},\zeta_{n_2},\zeta_{n_3}\rightarrow 0 \quad \text{as} \quad y\rightarrow\infty, \tag{55}$$

where $D\zeta=\partial\zeta/\partial y$. For a given ω, β_1, and R, one can solve (50)–(55) numerically to determine the complex eigenvalue α and the eigenfunctions ζ_{nm}.

Substituting (45)–(49) into (39)–(44) yields

$$\mathscr{L}_1(u_1,v_1,w_1)=-\sum_{n=1}^{2}\left(\frac{\partial A_n}{\partial x_1}\zeta_{n_1}+\frac{\partial A_n}{\partial z_1}\zeta_{n_3}\right)\exp(i\theta_n)-\sum_{n=1}^{2}A_n\frac{\partial\zeta_{n1}}{\partial x_1}\exp(i\theta_n), \tag{56}$$

$$\begin{aligned}\mathscr{L}_2(u_1,v_1,p_1)=&-\sum_{n=1}^{2}\left[\frac{\partial A_n}{\partial t_1}\zeta_{n_1}+\left(U_0\zeta_{n_1}+\zeta_{n_4}-\frac{2i\alpha}{R}\zeta_{n_1}\right)\frac{\partial A_n}{\partial x_1}+2i\beta_n\zeta_{n_1}\frac{\partial A_n}{\partial z_1}\right]\exp(i\theta_n)\\&-\sum_{n=1}^{2}\left[\left(U_0-\frac{2i\alpha}{R}\right)\frac{\partial\zeta_{n_1}}{\partial x_1}-\frac{i}{R}\zeta_{n_1}\frac{d\alpha}{dx_1}+\frac{\partial\zeta_{n_4}}{\partial x_1}+\frac{\partial U_0}{\partial x_1}\zeta_{n_1}+V_0\frac{\partial\zeta_{n_1}}{\partial y}\right]A_n\exp(i\theta_n)\\&-\tfrac{1}{2}\chi\left[iU_1\alpha\zeta_{11}+V_1\frac{\partial\zeta_{11}}{\partial y}+\frac{\partial U_1}{\partial y}\zeta_{12}-W_1\beta_1\zeta_{11}+2i\beta U_1\zeta_{13}\right]A_1\exp\\&\times[i\theta_2+2i(\beta-\beta_1)z_0]-\tfrac{1}{2}\chi\left[iU_1\alpha\zeta_{21}+V_1\frac{\partial\zeta_{21}}{\partial y}+\frac{\partial U_1}{\partial y}\zeta_{22}-W_1\beta_1\zeta_{21}-2i\beta U_1\zeta_{23}\right]\\&\times A_2\exp[i\theta_1-2i(\beta-\beta_1)z_0]+\text{NST},\end{aligned} \tag{57}$$

$$\begin{aligned}\mathscr{L}_3(v_1,p_1)=&-\sum_{n=1}^{2}\left[\frac{\partial A_n}{\partial t_1}\zeta_{n_2}+\left(U_0-\frac{2i\alpha}{R}\right)\zeta_{n_2}\frac{\partial A_n}{\partial x_1}+\frac{2i\beta_n}{R}\zeta_{n_2}\frac{\partial A_n}{\partial z_1}\right]\times\exp(i\theta_n)\\&-\sum_{n=1}^{2}\left[\left(U_0-\frac{2i\alpha}{R}\right)\frac{\partial\zeta_{n_2}}{\partial x_1}-\frac{i}{R}\zeta_{n_2}\frac{d\alpha}{\partial x_1}+\frac{\partial}{\partial y}(V_0\zeta_{n_2})\right]A_n\exp(i\theta_n)\\&-\tfrac{1}{2}\chi\left[i\alpha U_1\zeta_{12}+\frac{\partial}{\partial y}(V_1\zeta_{12})-\beta_1W_1\zeta_{12}+2i\beta V_1\zeta_{13}\right]A_1\exp[i\theta_2+2i(\beta-\beta_1)z_0]\\&-\tfrac{1}{2}\chi\left[i\alpha U_1\zeta_{22}+\frac{\partial}{\partial y}(V_1\zeta_{22})-\beta_1W_1\zeta_{22}-2i\beta V_1\zeta_{23}\right]\\&\times A_2\exp[i\theta_1-2i(\beta-\beta_1)z_0]+\text{NST},\end{aligned} \tag{58}$$

$$\mathscr{L}_4(w_1, p_1) = -\sum_{n=1}^{2}\left[\frac{\partial A_n}{\partial t_1}\zeta_{n3} + \left(U_0 - \frac{2i\alpha}{R}\right)\zeta_{n3}\frac{\partial A_n}{\partial x_1} + \left(\zeta_{n4} + \frac{2i\beta_n}{R}\zeta_{n3}\right)\frac{\partial A_n}{\partial z_1}\right]\exp(i\theta_n)$$
$$-\sum_{n=1}^{2}\left[\left(U_0 - \frac{2i\alpha}{R}\right)\frac{\partial \zeta_{n3}}{\partial x_1} - \frac{i}{R}\zeta_{n3}\frac{d\alpha}{dx_1} + V_0\frac{\partial \zeta_{n3}}{\partial y}\right]A_n\exp(i\theta_n)$$
$$-\tfrac{1}{2}\chi\left[iU_1\alpha\zeta_{13} + (2\beta-\beta_1)W_1\zeta_{13} + V_1\frac{\partial\zeta_{13}}{\partial y} - i\frac{\partial W_1}{\partial y}\zeta_{12}\right]A_1\exp[i\theta_2 + 2i(\beta-\beta_1)z_0]$$
$$-\tfrac{1}{2}\chi\left[iU_1\alpha\zeta_{23} + (2\beta-\beta_1)W_1\zeta_{23} + V_1\frac{\partial\zeta_{23}}{\partial y} + i\frac{\partial W_1}{\partial y}\zeta_{22}\right]$$
$$\times A_2\exp[i\theta_1 - 2i(\beta-\beta_1)z_0] + \mathrm{NST}, \tag{59}$$

where NST stands for terms that are proportional to $\exp[\pm i(\beta+\beta_1)z_0]$, which do not produce secular terms in u_1, v_1, w_1, and p_1.

Since the homogeneous parts of (56)–(59), (43), and (44) are the same as (33)–(38) and since the latter have a non-trivial solution, the inhomogeneous equations (56)–(59), (43), and (44) have a solution only if the inhomogeneous parts are orthogonal to every solution of the adjoint homogeneous problem (e.g. Nayfeh 1981). These solvability conditions depend on whether $\beta \approx \beta_1$ or not. If β is away from β_1, the solvability conditions yield two uncoupled equations describing the effect of non-parallelism on A_1 and A_2. If $\beta \approx \beta_1$, we introduce a detuning parameter σ defined by

$$\beta = \beta_1 + \epsilon\sigma, \tag{60}$$

where $\sigma = O(1)$ and express $(\beta-\beta_1)z_0$ as σz_1. Then imposing the solvability condition that the inhomogeneities be orthogonal to every solution of the adjoint homogeneous problem, we obtain

$$g_{11}\frac{\partial A_1}{\partial t_1} + g_{12}\frac{\partial A_1}{\partial x_1} + g_{13}\frac{\partial A_1}{\partial z_1} = h_1 A_1 + \chi h_{12}A_2\exp(-i\sigma z_1), \tag{61}$$

$$g_{21}\frac{\partial A_2}{\partial t_1} + g_{22}\frac{\partial A_2}{\partial x_1} + g_{23}\frac{\partial A_2}{\partial z_1} = h_2 A_2 + \chi h_{21}A_1\exp(i\sigma z_1), \tag{62}$$

where the g's and h's are given in the appendix together with the adjoint problems. Differentiating (33)–(38), respectively, with respect to α and β_n and imposing the solvability conditions, one can show that

$$\frac{g_{n2}}{g_{n1}} = \omega_\alpha, \quad \frac{g_{n3}}{g_{n1}} = \omega_{\beta_n} \tag{63}$$

where ω_α and ω_β are the complex group velocities in the x and z directions.

Since the solutions of (61) and (62) for general initial conditions are not available yet, we consider next the special case of a single-frequency disturbance that is perfectly tuned in the spanwise wavenumber. The single-frequency assumption corresponds to the case of a disturbance generated by a vibrating ribbon. The second assumption demands that $\beta = \beta_1$ and that the waves are modulated in the streamwise direction only. Thus, we consider the case in which $\partial A_n/\partial t_1 = \partial A_n/\partial z_1 = 0$ and $\sigma = 0$. Then, (61) and (62) can be rewritten as

$$\frac{dA_1}{dx} = \epsilon\hat{h}_1 A_1 + \epsilon_r\hat{h}_{12}A_2 \tag{64}$$

$$\frac{dA_2}{dx} = \epsilon \hat{h}_2 A_2 + \epsilon_v \hat{h}_{21} A_1, \tag{65}$$

where

$$\hat{h}_n = h_n/g_{n_2}, \quad \hat{h}_{12} = h_{12}/g_{12}, \quad \hat{h}_{21} = h_{21}/g_{22}.$$

It follows from (50)–(55) that $\zeta_{13} = -\zeta_{23}$ and $\zeta_{1_n} = \zeta_{2_n}$ for $n \neq 3$, while it follows from (A 7)–(A 12) that the adjoint solutions are related by $\zeta_{14}^* = -\zeta_{24}^*$ and $\zeta_{1_n}^* = \zeta_{2_n}^*$ for $n \neq 4$. Hence, it follows from (A 2), (A 4), (A 5), and (A 6) that

$$g_{12} = g_{22}, \quad h_1 = h_2, \quad h_{12} = h_{21}.$$

Thus,

$$\hat{h}_1 = \hat{h}_2 \quad \text{and} \quad \hat{h}_{12} = \hat{h}_{21}.$$

Therefore, adding (64) and (65) yields

$$\frac{d}{dx}(A_2 + A_1) = (\epsilon \hat{h}_1 + \epsilon_v \hat{h}_{12})(A_2 + A_1). \tag{66}$$

Subtracting (64) from (65) yields

$$\frac{d}{dx}(A_2 - A_1) = (\epsilon \hat{h}_1 - \epsilon_v \hat{h}_{12})(A_2 - A_1). \tag{67}$$

The solutions of (66) and (67) are

$$A_2 + A_1 = 2c_1 \exp\left[\int (\epsilon \hat{h}_1 + \epsilon_v \hat{h}_{12})\, dx\right], \tag{68}$$

$$A_2 - A_1 = 2c_2 \exp\left[\int (\epsilon \hat{h}_1 - \epsilon_v \hat{h}_{12})\, dx\right], \tag{69}$$

where c_1 and c_2, are arbitrary constants that can be determined from the initial conditions. Solving (68) and (69) gives

$$A_2 = c_1 \exp\left[\int (\epsilon \hat{h}_1 + \epsilon_v \hat{h}_{12})\, dx\right] + c_2 \exp\left[(\epsilon \hat{h}_1 - \epsilon_v \hat{h}_{12})\, dx\right], \tag{70}$$

$$A_1 = c_1 \exp\left[\int (\epsilon \hat{h}_1 + \epsilon_v \hat{h}_{12})\, dx\right] - c_2 \exp\left[(\epsilon \hat{h}_1 - \epsilon_v \hat{h}_{12})\, dx\right]. \tag{71}$$

Substituting for A_1 and A_2 in (45), using (49), substituting the results into (28), and recalling that $\beta_1 = \beta$, we obtain

$$\begin{aligned} u = \zeta_{11} \exp\left[\int (i\alpha + \epsilon \hat{h}_1)\, dx - i\beta z - i\omega t\right]\left[c_1 \exp\left(\epsilon_v \int \hat{h}_{12} dx\right) - c_2 \exp\left(-\epsilon_v \int \hat{h}_{12} dx\right)\right] \\ + \zeta_{21} \exp\left[\int (i\alpha + \epsilon \hat{h}_1)\, dx + i\beta z - i\omega t\right]\left[c_1 \exp\left(\epsilon_v \int \hat{h}_{12} dx\right)\right. \\ \left. + c_2 \exp\left(-\epsilon_v \int \hat{h}_{12} dx\right)\right] + \dots \end{aligned} \tag{72}$$

Equations (A 2) and (A 5) show that, in general, $\hat{h}_{12}$ is a complex number. Hence, one of the terms multiplying c_1 and c_2 decays whereas the other grows exponentially with distance. Thus, the growth rate σ based on u for either the wave with the positive or negative β is

$$\sigma = -\alpha_i + \epsilon\left[\mathrm{Re}\,(\hat{h}_1) + \frac{1}{\zeta_{11}} \frac{\partial \zeta_{11}}{\partial x_1}\right] + \epsilon_v |\mathrm{Re}\,(\hat{h}_{12})| \tag{73}$$

$F\times 10^{-6}$	α_r	$\sigma_{qp}\times 10^3$	$\sigma_v\times 10^3$
80	0·2094	−10·920	4·033
75	0·1993	−6·587	3·792
70	0·1884	−2·937	3·524
65	0·1769	0·006	3·279
60	0·1650	2·256	3·075
55	0·1527	3·843	2·918
50	0·1401	4·801	2·812
45	0·1273	5·165	2·757
40	0·1142	4·973	2·750
35	0·1009	4·268	2·781
30	0·0873	3·101	2·829
25	0·0735	1·549	2·864
21	0·0621	0·1105	2·887
20	0·0592	−0·2643	2·901
19	0·0562	−0·6410	2·923
18	0·0533	−1·017	2·960
17	0·0503	−1·389	3·015

TABLE 1. Comparison of $\sigma_{qp} = -\alpha_i$ with σ_v for $\epsilon_v = 0{\cdot}01$, $\beta = 0{\cdot}077463$, $R = 950$, $G_N = 13{\cdot}9566$.

because $\zeta_{11} = \zeta_{21}$. As $\epsilon_v \to 0$, (73) agrees with the non-parallel result of Nayfeh & Padhye (1979).

Equation (73) shows that the growth rate is the sum of three quantities: $\alpha_{qp} = -\alpha_i$, the quasi-parallel growth rate; $\sigma_{np} = \epsilon[\mathrm{Re}\,(\hat{h}_1) + \zeta_{11}^{-1}\,\partial\zeta_{11}/\partial x_1]$, the effect of non-parallelism; and $\sigma_v = \epsilon_v\,|\mathrm{Re}\,\hat{h}_{12}|$, the effect of the streamwise vortices. Thus, in a given physical situation, the relative influence of the vortices and non-parallelism depends on the relative magnitudes of ϵ and ϵ_v. For maximum amplified waves, $\epsilon = O(10^{-3})$, whereas for flows over concave surfaces, ϵ_v can be $O(0{\cdot}10)$, depending on the radius of curvature. In such situations, the effect of the vortices dominates the effect of non-parallelism, and the presence of the vortices is a very powerful instability mechanism. Hence, the presence of this mechanism may not be difficult to check experimentally.

4. Numerical results and discussion

We present numerical results for the case of flow past a cylinder with the streamwise cross-section in the form of a circular arc; in other words, the flow over a surface that has a constant curvature over a finite streamwise extent. For this case, the boundary-layer flow is given by the Blasius solution. In all results, the Reynolds number $R = 950$ and the Görtler number $G_N = 13{\cdot}9566$. For a given β, the computer code developed by Ragab & Nayfeh (1980) was used to calculate σ and U_1, V_1, W_1, and P_1 as functions of y. It was found that $\sigma = 1{\cdot}16225$ when $\beta = 0{\cdot}077436$, $\sigma = 1{\cdot}42879$ when $\beta = 0{\cdot}1$, and $\sigma = 1{\cdot}92067$ when $\beta = 0{\cdot}15$. As mentioned earlier, V_1, W_1, and P_1 are small compared with U_1. The three-dimensional code of Nayfeh & Padhye (1979) was used to determine α as a function of dimensionless frequency $F = \omega/R$ for the cases $\beta_1 = \pm\beta$. The results (α_r and $\sigma_{qp} = -\alpha_i$) are listed in columns 2 and 3 of tables 1–3 for $\beta = 0{\cdot}077436$, 0.1, and 0·15, respectively. Corresponding to each α, we calculated the ζ_{nm} and ζ_{nm}^*. Then using (A 2) and (A 5), we calculated g_{n_2} and h_{12} from which we calculated $\hat{h}_{12}$, and then $\sigma_v = \epsilon_v\,|\mathrm{Re}\,\hat{h}_1|$; it is listed in column 4 of tables 1–3.

$F \times 10^{-6}$	α_r	$\sigma_{qp} \times 10^3$	$\sigma_v \times 10^3$
80	0·2608	−12·844	6·047
75	0·1967	−8·339	5·668
70	0·1858	−4·517	5·290
65	0·1743	−1·409	4·962
60	0·1622	1·000	4·698
55	0·1498	2·743	4·501
50	0·1371	3·858	4·369
45	0·1241	4·384	4·297
40	0·1108	4·363	4·278
35	0·0973	3·841	4·303
30	0·0836	2·877	4·370
25	0·0691	1·551	4·509
21	0·0582	0·3150	4·756
20	0·0554	−0·0055	4·851
19	0·0524	−0·3276	4·963
18	0·0496	−0·6483	5·093
17	0·0466	−0·9651	5·243

TABLE 2. Comparison of $\sigma_{qp} = -\alpha_i$ with σ_v for $\epsilon_v = 0{\cdot}01$, $\beta = 0{\cdot}1$, $R = 950$, $G_N = 13{\cdot}9566$.

$F \times 10^{-6}$	α_r	$\sigma_{qp} \times 10^3$	$\sigma_v \times 10^3$
70	0·1787	−9·623	10·774
65	0·1671	−5·996	10·115
60	0·1550	−3·105	9·634
55	0·1424	−0·908	9·318
50	0·1295	0·645	9·148
45	0·1164	1·607	9·113
40	0·1030	2·029	9·201
35	0·0895	1·970	9·396
30	0·0759	1·498	9·667
25	0·0623	0·703	9·947
22	0·0542	0·118	10·068
21	0·0515	−0·087	10·090
20	0·0488	−0·295	10·100
19	0·0461	−0·506	10·096

TABLE 3. Comparison of $\sigma_{qp} = -\alpha_i$ with σ_v for $\epsilon_v = 0{\cdot}01$, $\beta = 0{\cdot}15$, $R = 950$, $G_N = 13{\cdot}9566$.

Tables 1–3 show that the presence of the vortices is destabilizing because it increases the range of frequencies that receive amplification and it increases the amplification rate at any frequency and spanwise wavenumber by a significant amount. The increase in the amplification rate increases with increasing spanwise wavenumber (i.e. decreasing spanwise wavelength). When $\beta = 0{\cdot}15$, the increase of the maximum amplification rate $2{\cdot}029 \times 10^{-3}$ is $9{\cdot}201 \times 10^{-3}$, which is more than fourfolds. Since a Görtler vortex having an amplitude of 1 % of the mean flow (i.e. $\epsilon_v = 0{\cdot}01$) is not uncommon, the present resonance instability mechanism may dominate the transition process.

It should be noted that the present analysis is valid only when the amplitude of the Tollmien–Schlichting waves ϵ_T is small compared with the amplitude of the vortices ϵ_v. As the Tollmien–Schlichting waves grow, one needs to account for their influence on the vortices. In fact, they will generate streamwise vortices having an amplitude $O(\epsilon_T^2)$ (Klebanoff, Tidstrom & Sargent 1962; Benney & Lin 1960; Antar & Collins 1975), which may strengthen or weaken the primary vortices, depending on their phasings. This effect has not been taken into account in this paper.

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Appendix

$$g_{n_1} = \int_0^\infty (\zeta_{n_1}\zeta_{n_2}^* + \zeta_{n_2}\zeta_{n_3}^* + \zeta_{n_3}\zeta_{n_4}^*)\,dy, \tag{A 1}$$

$$g_{n_2} = \int_0^\infty \left[\zeta_{n_1}\zeta_{n_1}^* + \zeta_{n_4}\zeta_{n_2}^* + \left(U_0 - \frac{2i\alpha}{R}\right)(\zeta_{n_1}\zeta_{n_2}^* + \zeta_{n_2}\zeta_{n_3}^* + \zeta_{n_3}\zeta_{n_4}^*)\right] dy, \tag{A 2}$$

$$g_{n_3} = \int_0^\infty \left[\zeta_{n_3}\zeta_{n_1}^* + \zeta_{n_4}\zeta_{n_3}^* + \frac{2i\beta_n}{R}(\zeta_{n_1}\zeta_{n_2}^* + \zeta_{n_2}\zeta_{n_3}^* + \zeta_{n_3}\zeta_{n_4}^*)\right] dy, \tag{A 3}$$

$$\begin{aligned} -h_n = \int_0^\infty \bigg\{ & \frac{\partial \zeta_{n_1}}{\partial x_1}\zeta_{n_1}^* + \left[\left(U_0 - \frac{2i\alpha}{R}\right)\frac{\partial \zeta_{n_1}}{\partial x_1} - \frac{i}{R}\zeta_{n_1}\frac{d\alpha}{dx_1} + \frac{\partial \zeta_{n_4}}{\partial x_1} + \frac{\partial U_0}{\partial x_1}\zeta_{n_1} + V_0\frac{\partial \zeta_{n_1}}{\partial y}\right]\zeta_{n_2}^* \\ & + \left[\left(U_0 - \frac{2i\alpha}{R}\right)\frac{\partial \zeta_{n_2}}{\partial x_1} - \frac{i}{R}\zeta_{n_2}\frac{d\alpha}{dx_1} + \frac{\partial}{\partial y}(V_0\zeta_{n_2})\right]\zeta_{n_3}^* \\ & + \left[\left(U_0 - \frac{2i\alpha}{R}\right)\frac{\partial \zeta_{n_3}}{\partial x_1} - \frac{i}{R}\zeta_{n_3}\frac{d\alpha}{dx_1} + V_0\frac{\partial \zeta_{n_3}}{\partial y}\right]\zeta_{n_4}^*\bigg\}\, dy, \end{aligned} \tag{A 4}$$

$$\begin{aligned} -2h_{12} = \int_0^\infty \bigg\{ & \left[i\alpha U_1\zeta_{21} + V_1\frac{\partial \zeta_{21}}{\partial y} + \frac{\partial U_1}{\partial y}\zeta_{22} - \beta_1 W_1\zeta_{21} - 2i\beta U_1\zeta_{23}\right]\zeta_{12}^* \\ & + \left[i\alpha U_1\zeta_{22} + \frac{\partial}{\partial y}(V_1\zeta_{22}) - \beta_1 W_1\zeta_{22} - 2i\beta V_1\zeta_{23}\right]\zeta_{13}^* \\ & + \left[i\alpha U_1\zeta_{23} + (2\beta - \beta_1)W_1\zeta_{23} + V_1\frac{\partial \zeta_{23}}{\partial y} + i\frac{\partial W_1}{\partial y}\zeta_{22}\right]\zeta_{14}^*\bigg\}\, dy, \end{aligned} \tag{A 5}$$

$$\begin{aligned} -2h_{21} = \int_0^\infty \bigg\{ & \left[i\alpha U_1\zeta_{11} + V_1\frac{\partial \zeta_{11}}{\partial y} + \frac{\partial U_1}{\partial y}\zeta_{12} - \beta_1 W_1\zeta_{11} + 2i\beta U_1\zeta_{13}\right]\zeta_{22}^* \\ & + \left[i\alpha U_1\zeta_{12} + \frac{\partial}{\partial y}(V_1\zeta_{12}) - \beta_1 W_1\zeta_{12} + 2i\beta V_1\zeta_{13}\right]\zeta_{23}^* \\ & + \left[i\alpha U_1\zeta_{13} + (2\beta - \beta_1)W_1\zeta_{13} + V_1\frac{\partial \zeta_{13}}{\partial y} - i\frac{\partial W_1}{\partial y}\zeta_{12}\right]\zeta_{24}^*\bigg\}\, dy. \end{aligned} \tag{A 6}$$

In the above the ζ_{nm}^* are solutions of the following adjoint problems:

$$i\alpha\zeta_{n_1}^* + i(U_0\alpha - \omega)\zeta_{n_2}^* - \frac{1}{R}(D^2 - \alpha^2 - \beta_n^2)\zeta_{n_2}^* = 0, \tag{A 7}$$

$$-D\zeta^*_{n_1} + DU_0\zeta^*_{n_2} + i(U_0\alpha-\omega)\zeta^*_{n_3} - \frac{1}{R}(D^2-\alpha^2-\beta^2_n)\zeta^*_{n_3} = 0, \tag{A 8}$$

$$-i\beta_n\zeta^*_{n_1} + i(U_0\alpha-\omega)\zeta^*_{n_4} - \frac{1}{R}(D^2-\alpha^2-\beta^2_n)\zeta^*_{n_4} = 0, \tag{A 9}$$

$$i\alpha\zeta^*_{n_2} - D\zeta^*_{n_3} - i\beta_n\zeta^*_{n_4} = 0, \tag{A 10}$$

$$\zeta^*_{n_2} = \zeta^*_{n_3} = \zeta^*_{n_4} = 0 \quad \text{at} \quad y = 0, \tag{A 11}$$

$$\zeta^*_{nm} \to 0 \quad \text{as} \quad y \to \infty. \tag{A 12}$$

REFERENCES

AIHARA, Y. 1962 *Bull. Aero. Res. Inst., Tokyo Univ.* **3**, 195.
AIHARA, Y. 1976 *Phys. Fluids* **19**, 1655.
ANTAR, B. N. & COLLINS, F. G. 1975 *Phys. Fluids* **18**, 289.
BENNEY, D. J. & LIN, C. C. 1960 *Phys. Fluids* **3**, 656.
BIPPES, H. 1978 *NASA* TM-75243.
BIPPES, H. & GÖRTLER, H. 1972 *Acta Mech.* **14**, 251.
BRADSHAW, P. 1965 *J. Fluid Mech.* **22**, 679.
CROW, S. C. 1966 *J. Fluid Mech.* **24**, 153.
FLORYAN, J. M. & SARIC, W. S. 1979 *A.I.A.A. Paper* no. 79–1497.
GREGORY, N. & WALKER, W. S. 1956 *ARC Rep.* M 2779.
GÖRTLER, H. 1941 *Z. angew. Math. Mech.* **21**, 250.
KLEBANOFF, P. S. & TIDSTROM, K. D. 1959 *N.A.S.A.* TN D 195.
KLEBANOFF, P. S., TIDSTROM, K. D. & SARGENT, L. M. 1962 *J. Fluid Mech.* **12**, 1.
NAYFEH, A. H. 1973 *Perturbation Methods*. Wiley.
NAYFEH, A. H. 1981 *Introduction to Perturbation Techniques*. Wiley.
NAYFEH, A. H. & PADHYE, A. 1979 *A.I.A.A. J.* **17**, 1084.
RAGAB, S. A. & NAYFEH, A. H. 1980 *A.I.A.A. Paper* no. 80–1377.
SMITH, A. M. O. 1955 *Quart. Appl. Math.* **13**, 233.
TANI, I. 1961 *Adv. Aero Sci.* **3**, 143.
TANI, I. 1962 *J. Geophys. Res.* **67**, 3075.
TANI, I. & AIHARA, Y. 1969 *Z. angew. Math. Phys.* **39**, 609.
TANI, I. & SAKAGAMI, J. 1962 *Proc. Int. Counc. Aero. Sci., Stockholm, Spartan*, p. 391.
WORTMANN, F. X. 1964 *Proc. XI Int. Cong. Appl. Mech.*, p. 815.
WORTMANN, F. X. 1969 *J. Fluid Mech.* **38**, 473.